Quantum Algorithms for Estimating Physical Quantities using Block-Encodings

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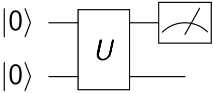
University of Texas at Austin

A review of modern techniques and new results from arXiv:2004.06832

May 2020

Quantum Primitives



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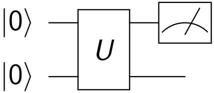

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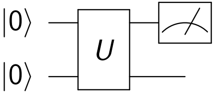

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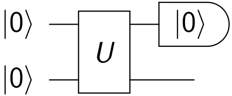

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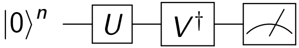

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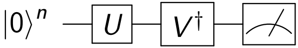

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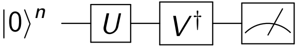

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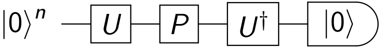

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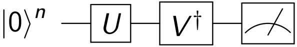

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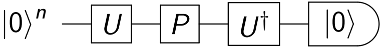

- Postselection probability: $|\langle\psi| P |\psi\rangle|^2$. Almost what we want.

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$$\left| \langle\psi| \frac{I+P}{2} |\psi\rangle \right| = \langle\psi| \frac{I+P}{2} |\psi\rangle = \frac{\langle\psi| P |\psi\rangle + 1}{2}$$

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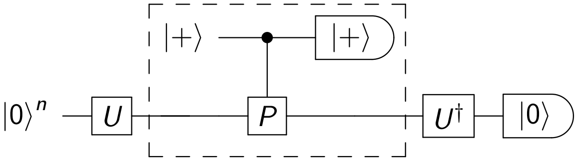

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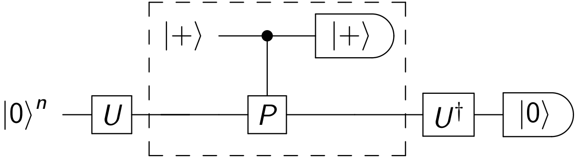

- Observe that $(\langle+| \otimes I)$CTRL-P$(|+\rangle \otimes I) = \frac{I+P}{2}$

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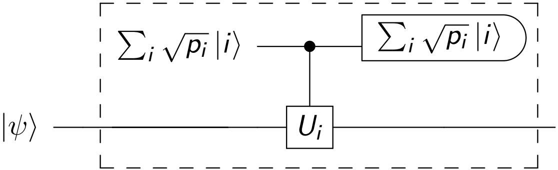

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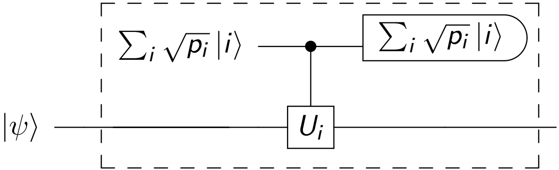

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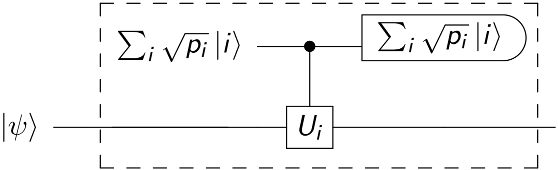

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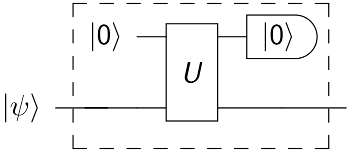
- Can 'perform' non-unitary operations!
  Berry, Childs, Kothari, Somma - arXiv:1501.01715, arXiv:1511.02306

Block-encodings

- Even more general form of circuit:



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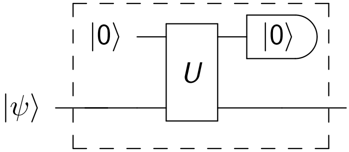

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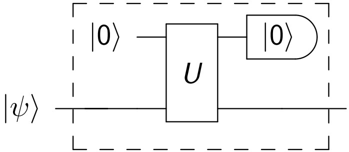

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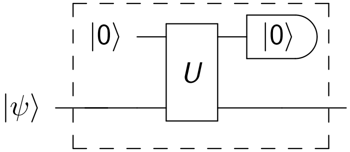

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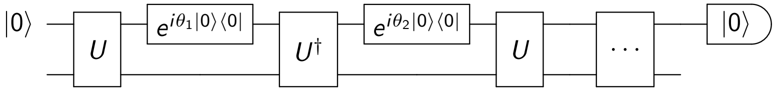

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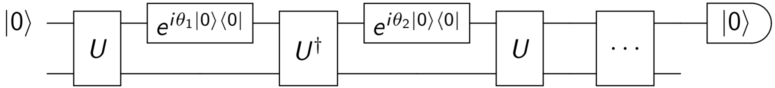

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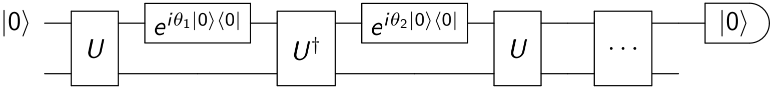

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  See arXiv:2003.02831.

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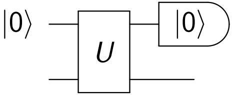

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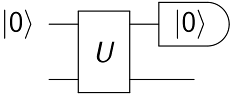

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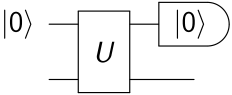

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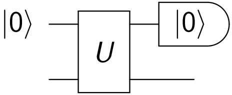

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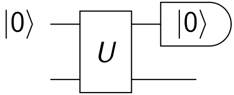

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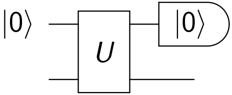

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- Say we have a Hamiltonian H with:
  - Spectral gap Δ
  - Known ground-state energy E_0
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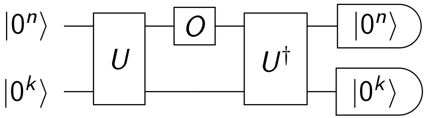

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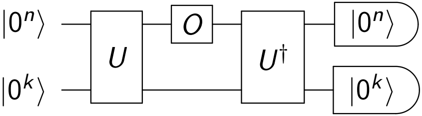

$$\rightarrow |\langle\psi| (O \otimes I) |\psi\rangle| = |\mathrm{Tr}(|\psi\rangle\langle\psi| O)| = |\mathrm{Tr}(\rho O)|$$

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- Improves over Pedernales et al. arXiv:1401.2430

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- Usually interested in histograms or 'sketches' of $\rho(E)$

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- Histogram bin: $\int_{E_a}^{E_b} \rho(E) dE$

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- Both improve upon phase-estimation method.

Kernel Polynomial Method (this work)

- Method for sketching $\rho(E)$: Chebyshev decomposition

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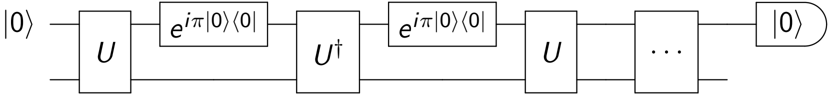
- Quantum singular value transformation makes $T_n(H)$ very easy:



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- Correlation functions in linear response theory:
  - Some observables B, C.

$$A(E) = \langle B\delta(H - E - E_0)C\rangle$$

Thank you for your attention!

Special thanks to: Scott Aaronson, Andras Gilyén, Andrew Potter, Justin Thaler, Chunhao Wang, Alexander Weisse and Alexandro Roggero

Oblivious amplitude amplification

- Primitive operation:

$$|\psi\rangle \to \frac{A|\psi\rangle}{|A|\psi\rangle|}$$

- Requires $O\left(\frac{1}{|A|\psi\rangle|}\right)$ applications of both A and preparations of $|\psi\rangle$
- What if $|\psi\rangle$ is very expensive to prepare?
- Oblivious amplitude amplification arXiv:1312.1414
- If A is (approximately) unitary, need exactly one copy of $|\psi\rangle$