"Improved classical simulation of quantum circuits dominated by Clifford gates"

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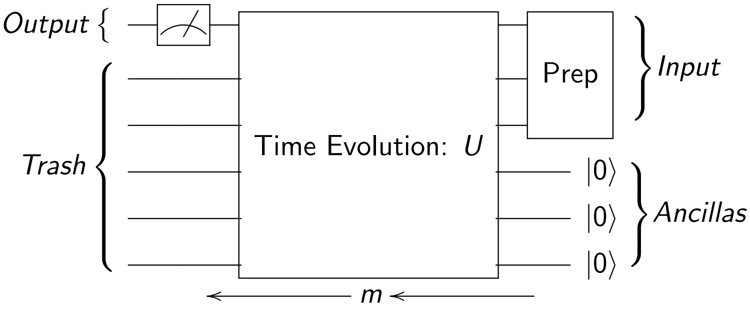
Model of Computation



- System: input qubits + ancilla qubits = n qubits
- Hilbert space: $\mathbb{C}^{2n}$ - exponentially large
- Intuition: Quantum computation efficiently calculates matrix multiplication $U|\Psi_{in}\rangle$

Model of Computation: Output Probabilities

- Simplification: assume input state $\left|0^{\otimes n}\right\rangle$
- Measure output qubit: some probability distribution
- Goal: sample from this distribution

$$c_x = \left\langle x \middle| U \middle| 0^{\otimes n} \right\rangle$$

$$P_x = |c_x|^2 = \left\langle 0^{\otimes n} \middle| U^\dagger |x\rangle\langle x| U \middle| 0^{\otimes n} \right\rangle$$

- Here $|x\rangle\langle x|$ is a projector onto an output state

Classical simulation: naïve approach

- Given $U = U_m \ldots U_3 U_2 U_1$, calculate P_x:

$$P_x = \langle 0^{\otimes n} \big| U_1^\dagger U_2^\dagger \ldots U_m^\dagger | x \rangle \langle x | U_m \ldots U_2 U_1 \big| 0^{\otimes n} \rangle$$

- Calculate m matrix multiplications in $\mathbb{C}^{2n}$
- Naïve runtime: $m(2^n)^\omega$, best known[1] $\omega = 2.3737$

Interpretation

- Exponential in n: always intractable for large enough n
- Getting rid of exponentiality? Would imply:
  Quantum computing = Classical computing
- Algorithm 'moves' exponent in n to other parameter

[1] Coppersmith-Winograd algorithm

Stabilizer States

- D. Gottesman (1998): "The Heisenberg Representation of Quantum Computers"[2]
- Consider an Abelian subgroup $G \subset \mathcal{P}_n$ with $-I \notin G$.
- Def: $|\phi\rangle$ is **stabilized** by G if $P|\phi\rangle = |\phi\rangle, \forall P \in G$.
- Def: $|\phi\rangle$ is a **stabilizer state** if stabilized by some G
- Clifford gates map stabilizer states to stabilizer states

- Example: $G = \langle X \otimes X, Z \otimes Z \rangle \subset \mathcal{P}_2$
- Unique stabilizer state:

$$|\phi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

- Degrees of freedom: G defined by k **stabilizer generators** stabilizes 2^{n-k} states.

[2] https://arxiv.org/abs/quant-ph/9807006v1

The Gottesmann-Knill Theorem

- Stabilizer projector: project onto space stabilized by G

$$\Pi_G = \prod_{P \in G} \frac{1+P}{2}$$

- Clifford gates can act on stabilizer projectors: act on each generator of G
- Want to calculate:

$$P_x = \left\langle 0^{\otimes n} \middle| U_1^\dagger U_2^\dagger \ldots U_m^\dagger |x\rangle\langle x| U_m \ldots U_2 U_1 \middle| 0^{\otimes n} \right\rangle$$

- What if $U_i \in \mathcal{C}_n$? Then, given $|x\rangle\langle x| = \Pi_x$:

$$P_x = \left\langle 0^{\otimes n} \middle| U_1^\dagger U_2^\dagger \ldots \Pi_{G_m(x)} \ldots U_2 U_1 \middle| 0^{\otimes n} \right\rangle$$

$$= \left\langle 0^{\otimes n} \middle| U_1^\dagger \Pi_{G_2(x)} U_1 \middle| 0^{\otimes n} \right\rangle = \left\langle 0^{\otimes n} \middle| \Pi_{G(x)} \middle| 0^{\otimes n} \right\rangle$$

- Result: can calculate circuit in polynomial time!

Clifford + T: A universal gate set

- Circuits composed of Cliffords, i.e. $H, S, CNOT$, can be simulated efficiently
- $\{H, S, CNOT\}$ **acting on stabilizer states only** is not universal
  - Mathematical fact
  - Otherwise: quantum computing = classical computing
- Add the T gate to obtain universal gate set:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

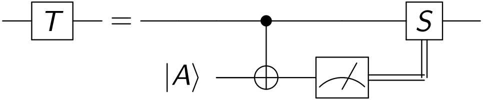
- Problem: T gate hard to simulate classically
- Incidentally: T gate also hard to build in experiment

Gadgetization with Magic States

- Consider a ‘magic state’:

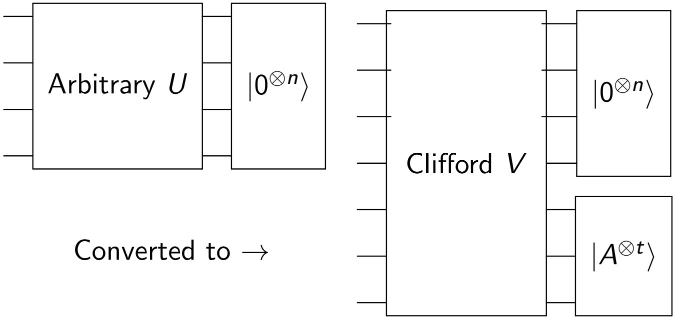
$$|A\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$$

- Use $|A\rangle$ as a resource to write T in terms of Cliffords:



- Measurement destroys magic state in the process.
- Input to circuit was $|0^{\otimes n}\rangle$, now is $|0^{\otimes n}A^{\otimes t}\rangle$.
- t = number of T gates in circuit

Converting the problem



The challenge

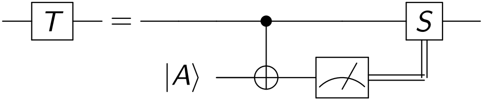
- Before: non-Clifford circuit with T gates
- After: non-stabilizer 'magic' resource state $|A^{\otimes t}\rangle$

Algorithm Goal

- Goal: Sample from probability distribution

$$P_x = \langle 0^{\otimes n} | U^\dagger \Pi_X U | 0^{\otimes n} \rangle$$

- Gadgetize non-Clifford unitary U to Clifford V with 'magic' resource state $|A^{\otimes t}\rangle$



- How to deal with measurement? Post-select measurement outcomes into string y. Calculate:

$$P_x = \frac{\langle A^{\otimes t} 0^{\otimes n} | V^\dagger (\Pi_x \otimes \Pi_y) V | 0^{\otimes n} A^{\otimes t} \rangle}{\langle A^{\otimes t} 0^{\otimes n} | V^\dagger (\mathbb{I} \otimes \Pi_y) V | 0^{\otimes n} A^{\otimes t} \rangle}$$

- Works for any y!

Concept: Stabilizer Rank χ

- Remaining problem: non-stabilizer state $|A^{\otimes t}\rangle$
- Write as a linear combination of χ stabilizer states $|\phi_a\rangle$

$$\left|A^{\otimes t}\right\rangle \approx \sum_{a}^{\chi} z_a |\phi_a\rangle = |\Psi\rangle$$

- 2^t stabilizer states: Naïve upper bound $\chi \leq 2^t$
- Clever trick 1: Recognize that $\left|A^{\otimes 2}\right\rangle$ is a sum of two stabilizer states. Divide $|A^{\otimes t}\rangle$ into pairs: $\chi \leq 2^{t/2}$
- Clever trick 2 (see appendix): Achieve $\chi \sim O(2^{0.23t})$. Authors conjecture that this is optimal.

Summary Calculation

$$P_x = \left\langle 0^{\otimes n} \middle| U^\dagger \Pi_x U \middle| 0^{\otimes n} \right\rangle$$

$$= \frac{\langle A^{\otimes t} 0^{\otimes n} | V^\dagger (\Pi_x \otimes \Pi_y) V | 0^{\otimes n} A^{\otimes t} \rangle}{\langle A^{\otimes t} 0^{\otimes n} | V^\dagger (\mathbb{I} \otimes \Pi_y) V | 0^{\otimes n} A^{\otimes t} \rangle}$$

$$= \frac{\langle A^{\otimes t} 0^{\otimes n} | \Pi_{G(x,y)} | 0^{\otimes n} A^{\otimes t} \rangle}{\langle A^{\otimes t} 0^{\otimes n} | \Pi_{H(y)} | 0^{\otimes n} A^{\otimes t} \rangle} = \frac{1}{2^u} \frac{\langle A^{\otimes t} | \Pi_{\bar{G}(x,y)} | A^{\otimes t} \rangle}{\langle \bar{A}^{\otimes t} \middle| \Pi_{H(y)} \middle| A^{\otimes t} \rangle}$$

$$= \frac{1}{2^u} \frac{|\Pi_{\bar{G}(x,y)} | A^{\otimes t} \rangle|^2}{|\Pi_{\bar{H}(y)} | A^{\otimes t} \rangle|^2} \approx \frac{1}{2^u} \frac{|\Pi_{\bar{G}(x,y)} | \Psi \rangle|^2}{|\Pi_{\bar{H}(y)} | \Psi \rangle|^2}$$

- Calculation boils down to $|\Pi_{\bar{G}(x,y)}|\Psi\rangle|^2$ and $|\Pi_{\bar{H}(y)}|\Psi\rangle|^2$
- Approx. requires random y, rather than arbitrary y

The Algorithm

1. Choose random y, evaluate projectors $\Pi_{\bar{G}(x,y)}$, $\Pi_{\bar{H}(y)}$
2. Compute $|A^{\otimes t}\rangle \approx \sum_a^{\chi} z_a|\phi_a\rangle = |\Psi\rangle$
   such that $|\langle A^{\otimes t}|\Psi\rangle|^2 \geq 1-\delta$,
   where $|\phi_a\rangle$ are stabilizer states, $\chi = O(2^{0.23t}\delta^{-1})$
3. Evaluate inner products $\left|\Pi_{\bar{G}(x,y)}|\Psi\rangle\right|^2$ and $\left|\Pi_{\bar{H}(y)}|\Psi\rangle\right|^2$

$$|\Pi|\Psi\rangle|^2 = \left|\Pi\sum_a^{\chi} z_a|\phi_a\rangle\right|^2 = \left|\sum_a^{\chi} z_a\Pi|\phi_a\rangle\right|^2$$

4. Compute distribution $P_{x=0}, P_{x=1} = 1 - P_{x=0}$, and sample from distribution.

Runtime

- Sample from output distribution for a string x:

$$\text{poly}(n, m) + 2^{0.23t} t^3 w^4$$

  - Exponential: number of T gates t
  - Polynomial: n qubits, width m circuit
  - Length of output string $|x| = w$
- Projector Π_x has 2^w generators: w^4 via trick (appendix)
- Exponential part is highly parallelizable
  - Each term in $\sum_a^{\chi} z_a \Pi |\phi_a\rangle$ can be calculated independently

Conclusions, next steps

Implementation

- MATLAB implementation by Bravyi, Gosset
  - Hidden shift algorithm on a laptop
  - 40 qubits, 50 T gates
- Python+C implementation by Iskren Vankov, me
- Upcoming: CUDA implementation?

New concept: Stabilizer Rank

- How to decompose arbitrary $|\Psi\rangle$ into stabilizer states?
- Improve naïve runtime $O(m2^n)$ to $O(m2^{\alpha n})$ for $\alpha < 1$?

Appendix

- Alexei Kitaev's Stabilizer Toolkit
- Sampling larger bitstrings x
- Stabilizer decomposition of $|A^{\otimes t}\rangle$
- Computing inner products in $O(\chi)$ rather than $O(\chi^2)$

Alexei Kitaev's Stabilizer Toolkit

- Traditional representation: $G \subset \mathcal{P}_n$
- Efficient representation:[3]
  - Affine space $\mathcal{K}$: Subspace of $\mathbb{F}^2$ such that $\mathcal{L}(\mathcal{K}) = h \oplus \mathcal{K}$
  - Quadratic form q: Function $q : \mathcal{K} \to \mathbb{Z}_8$ with properties

$$|\mathcal{K}, q\rangle = 2^{-k/2} \sum_{x \in \mathcal{K}} e^{\frac{i\pi}{4} q(x)} |x\rangle$$

- Algorithms:
  - Inner product of two states in $O(n^3)$
  - Measure a Pauli operator in $O(n^2)$
  - Sample random stabilizer states in $O(n^2)$ on average ($O(n^3)$ worst case)

[3] H. J. Garca and I. L. Markov, "Hybrid Techniques for Simulating Quantum Circuits using the Heisenberg Representation"

Sampling larger bitstrings x

- Projector Π_x has 2^w generators. Contributes to $|\Pi|\Psi\rangle|^2$.
- Achieve polynomial time: Sample first bit of x, x_1, then evaluate conditional probability for next bit, etc:

$$x = x_1 x_2 \ldots x_w$$

$$P(x_2|x_1) = \frac{P(x_1, x_2)}{P(x_1)} \rightarrow P(x_3|x_1 x_2) = \frac{P(x_1, x_2, x_3)}{P(x_1, x_2)} \ldots$$

- w^4: not particularly fast, but at least not exponential

Stabilizer decomposition of $|A^{\otimes t}\rangle$

$$|A^{\otimes t}\rangle \approx \sum_a^{\chi} z_a|\phi_a\rangle = |\Psi\rangle$$

$$|\langle A^{\otimes t}|\Psi\rangle|^2 \geq 1-\delta$$

- Low stabilizer rank decomposition $\chi = O(2^{0.23t}\delta^{-1})$

$$|A\rangle = e^{i\pi/8}HS^\dagger\left(\cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle\right) = e^{i\pi/8}HS^\dagger|H\rangle$$

- Find $|\mathcal{L}\rangle$ such that $|\langle H^{\otimes t}|\mathcal{L}\rangle|^2 \geq 1-\delta$
- Choose a dimension k such that $4 \geq 2^k v^{2t}\delta \geq 2$, $v = \cos\frac{\pi}{8}, |H\rangle = \frac{1}{2v}(|0\rangle + |+\rangle)$
- Sample a random subspace $\mathcal{L} \subset \mathbb{F}_2^n$ with dimension k. From Markov's inequality:

$$\Pr\left[|\langle H^{\otimes t}|\mathcal{L}\rangle|^2 \geq 1-\delta\right] \geq \Omega(\delta)$$

- Keep sampling $\mathcal{L}$ until this is true. Should take $O(\delta^{-1})$.

Computing inner products in $O(\chi)$ rather than $O(\chi^2)$

- Given $\Pi|\Psi\rangle = \sum_a^\chi z_a \Pi|\phi_a\rangle$, compute $|\Pi|\Psi\rangle|^2$
- Naïve method: $O(\chi^2)$ calculations of $z_a z_b \langle\phi_a|\Pi|\phi_b\rangle$
- With clever trick: $O(\chi)$ calculations of $z_a\langle\theta_i|\Pi|\phi_a\rangle$ with $L = 1/p_f\epsilon$ random states $|\theta_i\rangle$

$$\text{Random variable } \alpha = \frac{2^t}{L}\sum_{i=1}^{L} |\langle\theta_i|\Pi|\Psi\rangle|^2$$

$$\Pr\left[(1-\epsilon)|\Pi|\Psi\rangle|^2 \leq \alpha \leq (1+\epsilon)|\Pi|\Psi\rangle|^2\right] \geq 1 - p_f$$

- Derivation uses that stabilizer states $\mathcal{S}$ are a 2-design

$$\sum_{\theta\in\mathcal{S}} (|\theta\rangle\langle\theta| \otimes |\theta\rangle\langle\theta|) = \int_{\text{Haar}} (|\phi\rangle\langle\phi| \otimes |\phi\rangle\langle\phi|) d\phi$$