Amplitude Estimation based on Quantum Signal Processing

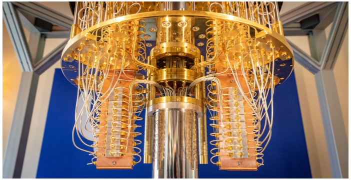
Patrick Rall, Bryce Fuller

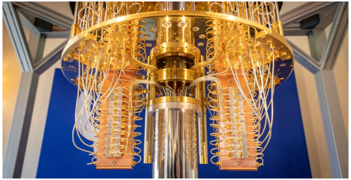
arXiv:2207.08628

IBM **Quantum**

September 2022

- Digital computation with superposition
- Applications: physics, chemistry, cryptography, finance...
- IBM Eagle (2021): 127 Quantum Bits





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- Warm-up: probabilistic computation

$$\text{States: } \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1/2\\1/2\end{bmatrix}, \begin{bmatrix}2/3\\1/3\end{bmatrix} \rightarrow \vec{p} \in \mathbb{R}_+^{2^n}, \ |\vec{p}|_1 = 1$$

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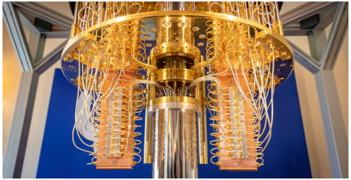

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A Short Intro to QC

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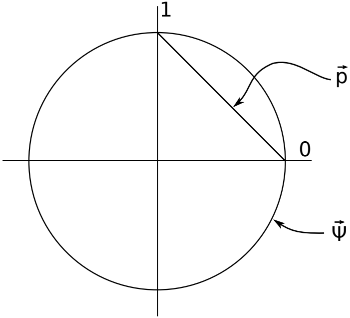

Grover's algorithm

- All dynamics take place in 2d subspace span$(\vec{\psi}, \vec{\phi}) \in \mathbb{C}^{2^n}$

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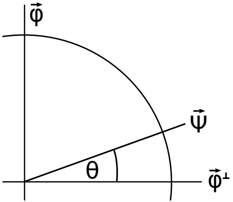

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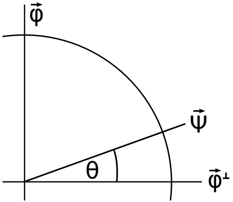

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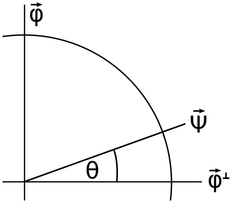

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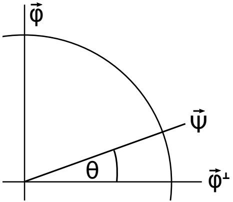

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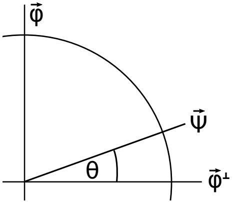

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  - Applications: Monte Carlo estimation, partition function estimation

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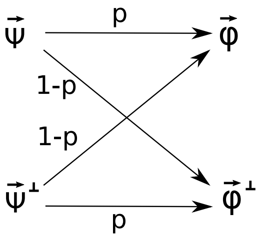

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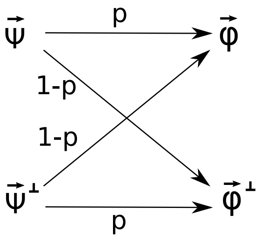

Quantum Signal Processing

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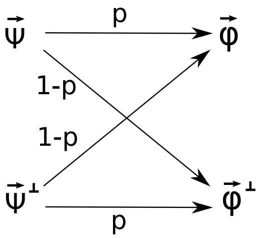

- Goal: estimate $a = |\vec{\psi}^* \vec{\phi}|$ in as few operations as possible.

Puzzle: estimating a via polynomials

Say $a \in [0,1]$ is unknown. For polynomials $p : [0,1] \to [0,1]$, you can toss a coin with bias $p(a)^2$ at cost $\deg(p)$.
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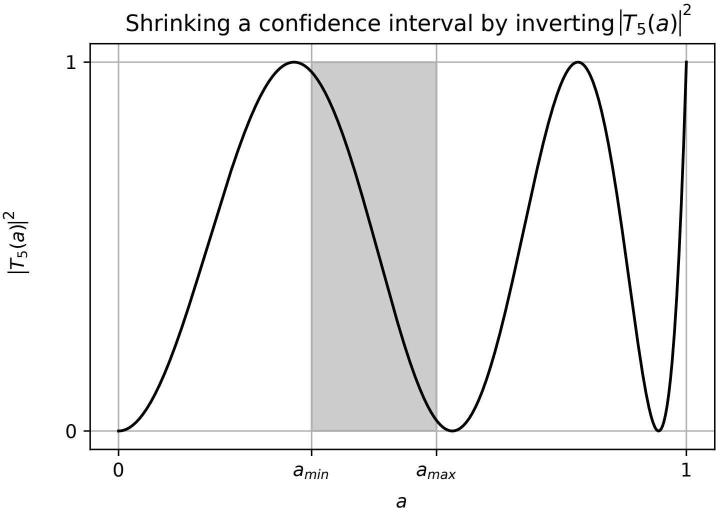
3. Return the midpoint of $[a_{\text{min}}, a_{\text{max}}]$.

Fast estimation: Interval refinement



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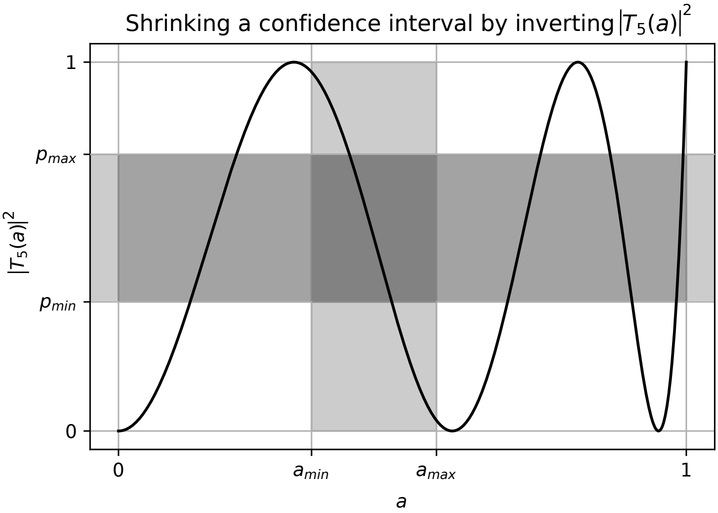

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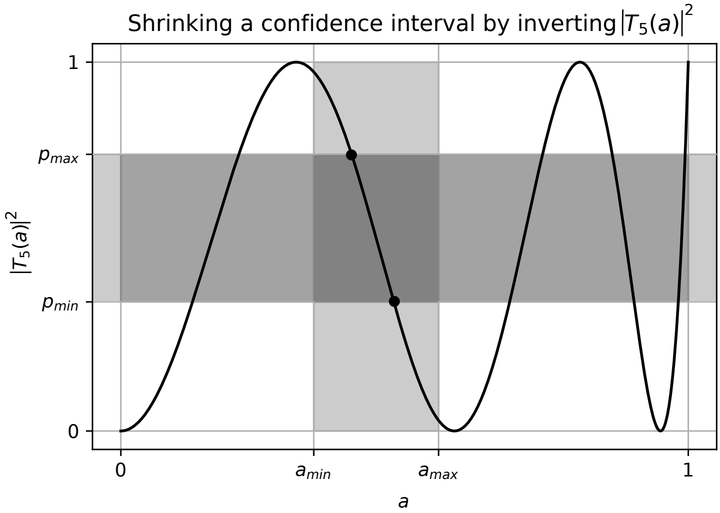

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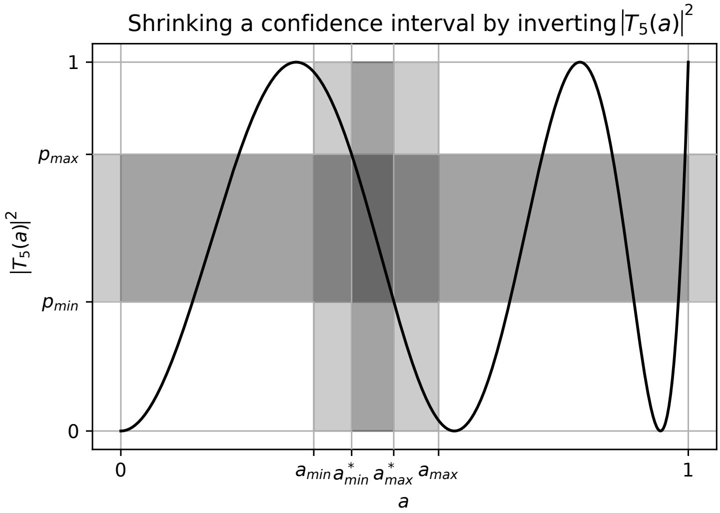

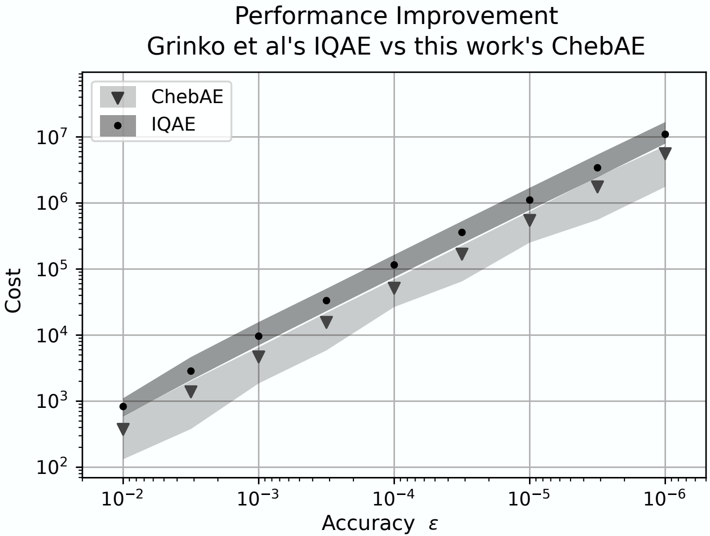
Performance Improvement
Grinko et al's IQAE vs this work's ChebAE
ChebAE
IQAE
Cost
10^2
10^3
10^4
10^5
10^6
10^7
10^{-2}
10^{-3}
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Accuracy ε

Results

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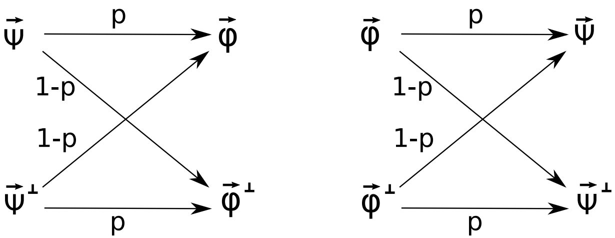

Non-destructive Estimation

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Lemma: post-algorithm state

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Then, if $a < \sqrt{\delta}/D$ then with probability $\geq 1 - \delta$ we have either $\vec{\psi}$ or $\vec{\phi}^{\perp}$.

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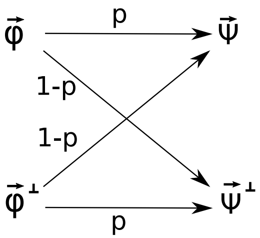
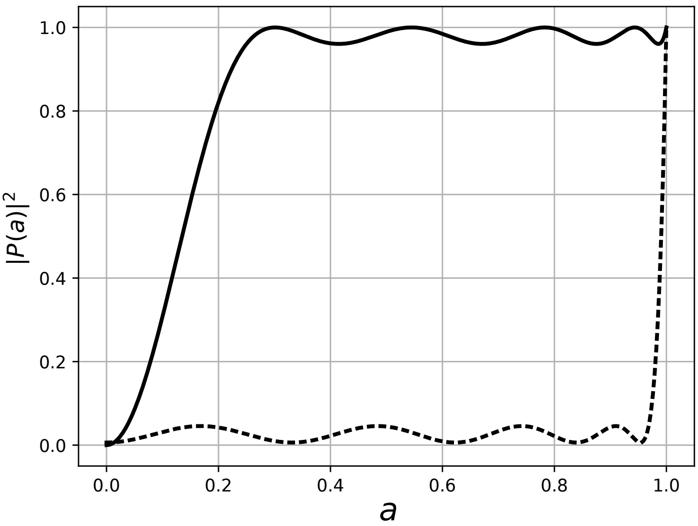
- If we have $\vec{\phi}$, we can assume $a > \sqrt{\delta}/D$.
- If we have $\vec{\phi}^{\perp}$, we leverage $a < 1/2$.

Converting $\vec{\phi}, \vec{\phi}^\perp \to \vec{\psi}$





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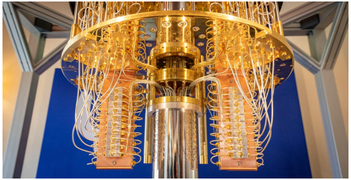
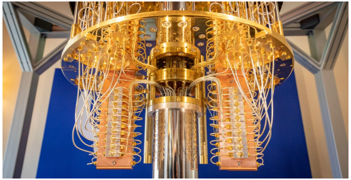
Hybrid quantum-classical estimation

- Current quantum computers are noisy
  $\to$ can only run programs of a certain length
- Program length $\sim$ polynomial degree
- Can think of degree $O(1)$ as classical limit



Hybrid quantum-classical estimation



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  $\rightarrow$ can only run programs of a certain length
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		Max Degree	Total Degree
Quantum		$O(1/\varepsilon)$	$O(1/\varepsilon)$
Classical		$O(1)$	$O(1/\varepsilon^2)$
Hybrid	$\beta \in [0, 1]$	$O(1/\varepsilon^{1-\beta})$	$O(1/\varepsilon^{1+\beta})$

Problem was first posed by Giurgica-Tironc et al 2012.03348.

Method:

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2. While $a_{\text{max}} - a_{\text{min}} > 2\varepsilon$:

Amplitude estimation via interval refinement

Method:

1. Initialize a confidence interval $[a_{\text{min}}, a_{\text{max}}] = [0, 1]$
2. While $a_{\text{max}} - a_{\text{min}} > 2\varepsilon$:
   1. Let $\Delta = a_{\text{max}} - a_{\text{min}}$. Decide if:

$$a \in [a_{\text{min}} + 0.1\Delta, a_{\text{max}}] \quad \text{(case I)}$$

$$a \in [a_{\text{min}}, a_{\text{max}} - 0.1\Delta] \quad \text{(case II)}$$

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   2. If I, set a_{min} to $a_{\text{min}} + 0.1\Delta$. If II set a_{max} to $a_{\text{max}} - 0.1\Delta$.

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   2. If I, set a_{min} to $a_{\text{min}} + 0.1\Delta$. If II set a_{max} to $a_{\text{max}} - 0.1\Delta$.
3. Return the midpoint of $[a_{\text{min}}, a_{\text{max}}]$.

Interval refinement with limited degree

- Decide using $O(1/\Delta^{1-\beta})$ maximum degree and $O(1/\Delta^{1+\beta})$ total degree:

$$a \in [a_{\text{min}} + 0.1\Delta, a_{\text{max}}] \quad \text{(case I)}$$

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Interval refinement with limited degree

IBM **Quantum**

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Interval refinement with limited degree

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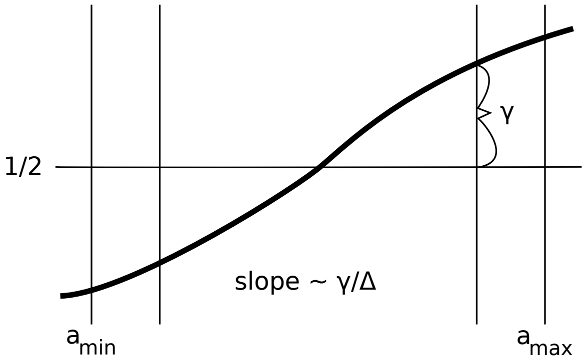
- Idea: Combine classical amplification with quantum polynomial degree.
- Say we can build a polynomial $p(a)$ such that:

$$a \leq a_{\min} + 0.1\Delta \;\; \rightarrow \;\; p(a) \leq \frac{1}{2} - \gamma$$

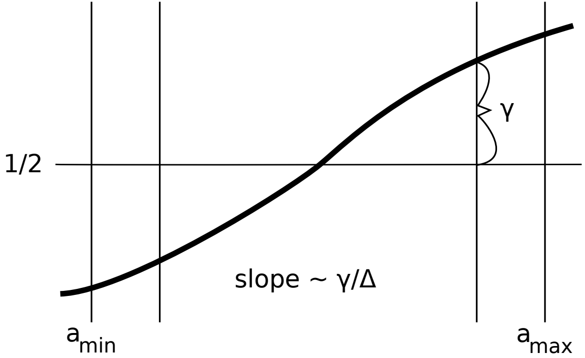
$$a \geq a_{\max} - 0.1\Delta \;\; \rightarrow \;\; p(a) \geq \frac{1}{2} + \gamma$$

$\rightarrow$ then we can distinguish I vs II in $O(1/\gamma^2)$ tries.



1/2
γ
slope ∼ γ/Δ
a_{min}
a_{max}

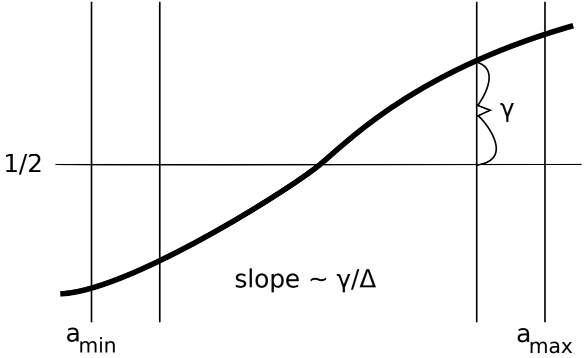
Crafting a polynomial for interval refinement



Polynomial degree $\sim$ slope: $O\left(\frac{\gamma}{\Delta}\right)$ Total cost: $O\left(\frac{1}{\gamma^2}\frac{\gamma}{\Delta}\right)$.

Crafting a polynomial for interval refinement

IBM **Quantum**



Polynomial degree $\sim$ slope: $O\left(\frac{\gamma}{\Delta}\right)$ Total cost: $O\left(\frac{1}{\gamma^2}\frac{\gamma}{\Delta}\right)$.

Achieve $O(1/\Delta^{1-\beta})$ maximum degree and $O(1/\Delta^{1+\beta})$ total degree by selecting $\gamma = \Delta^\beta$.

Puzzle: estimating a via polynomials

Say $a \in [0,1]$ is unknown. For polynomials $p : [0,1] \to [0,1]$, you can toss a coin with bias $p(a)^2$ at cost $\deg(p)$.
Estimate a to precision ε while minimizing cost.

1. Fast estimation: Estimate a to precision ε in $\approx 1.7/\varepsilon$ ✓
2. Non-destructive estimation: given exactly one input state $\vec{\psi}$ ✓
3. Hybrid quantum-classical estimation: cost $O(1/\varepsilon^{1+\beta})$, but maximum polynomial degree $O(1/\varepsilon^{1-\beta})$ ✓
4. Unbiased estimation: output an estimate $\hat{a}$ with $\mathbb{E}[\hat{a}] \approx a$

Unbiased estimation

Goal: sample from a random variable $\hat{a}$ such that:

$$\mathbb{E}[\hat{a}] \approx a \quad \textit{and} \quad \Pr[|\hat{a} - a| > \varepsilon] \leq \delta$$

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Approach:

- Say $a \in [a_{\text{min}}, a_{\text{max}}]$ such that $a_{\text{max}} - a_{\text{min}} < \varepsilon$. Pick $\hat{a} = a_{\text{max}}$ with probability

$$\frac{a - a_{\text{min}}}{a_{\text{max}} - a_{\text{min}}},$$

otherwise pick $\hat{a} = a_{\text{min}}$. Then $\mathbb{E}[\hat{a}] = a$ and $|\hat{a} - a| < \varepsilon$.

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- Construct a polynomial $p(a)$ such that $p(a)^2 \approx \frac{a - a_{\text{min}}}{a_{\text{max}} - a_{\text{min}}}$. Then $\mathbb{E}[\hat{a}] \approx a$.

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- Construct a polynomial $p(a)$ such that $p(a)^2 \approx \frac{a - a_{\text{min}}}{a_{\text{max}} - a_{\text{min}}}$. Then $\mathbb{E}[\hat{a}] \approx a$.
- Apply a recursive argument to the interval refinement algorithm: if all intermediate $\hat{a}$ satisfy $\mathbb{E}[\hat{a}] \approx a$, then so does the algorithm as a whole.

Thank you for your attention!

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Stefan Woerner, Julien Gacon, and Giacomo Nannicini
John Martyn, Nikitas Stamatopoulos, and Pawel Wocjan

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